Question 1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

(a) $\int_0^6 f(t)\,dt = 142.274$ or 142.275 cubic feet	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) Rate of change is $f(8) - g(8) = -59.582$ or -59.583 cubic feet per hour.	1 : answer
(c) $h(0) = 0$ For $0 < t \le 6$, $h(t) = h(0) + \int_0^t g(s)\,ds = 0 + \int_0^t 0\,ds = 0.$ For $6 < t \le 7$, $h(t) = h(6) + \int_6^t g(s)\,ds = 0 + \int_6^t 125\,ds = 125(t-6).$ For $7 < t \le 9$, $h(t) = h(7) + \int_7^t g(s)\,ds = 125 + \int_7^t 108\,ds = 125 + 108(t-7).$ Thus, $h(t) = \begin{cases} 0 & \text{for } 0 \le t \le 6 \\ 125(t-6) & \text{for } 6 < t \le 7 \\ 125 + 108(t-7) & \text{for } 7 < t \le 9 \end{cases}$	$3 : \begin{cases} 1 : h(t) \text{ for } 0 \le t \le 6 \\ 1 : h(t) \text{ for } 6 < t \le 7 \\ 1 : h(t) \text{ for } 7 < t \le 9 \end{cases}$
(d) Amount of snow is $\int_0^9 f(t)\,dt - h(9) = 26.334$ or 26.335 cubic feet.	$3 : \begin{cases} 1 : \text{integral} \\ 1 : h(9) \\ 1 : \text{answer} \end{cases}$

CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

1A1

A graphing calculator is required for some problems or parts of problems.

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Work for problem 1(a)

$\int_0^6 f(t)\,dt$

$\int_0^6 7te^{\cos t}\,dt$

$= 142.275 \text{ ft}^3$

Work for problem 1(b)

$f(t) - g(t)$ at 8 am

$7te^{\cos t} - 108$ cubic feet per hour

$7(8)e^{\cos 8} - 108$

$= -59.583 \text{ ft}^3/\text{hr}$

Continue problem 1 on page

Work for problem 1(c)

$$h(t) = \begin{cases} 0 & \text{for } 0 \le t \le 6 \\ 125(t-6) & \text{for } 6 < t \le 7 \\ 108(t-7)+125 & \text{for } 7 < t \le 9 \end{cases}$$

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Work for problem 1(d)

$$\int_0^9 f(t)\,dt - \int_0^9 g(t)\,dt$$

$$\int_0^9 7te^{\cos t}\,dt - h(t)\Big|_0^9$$

$$367.334 - (125 + 216)$$

$= 26.334\ ft^3$ of snow are on the driveway at 9 am.

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CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

1B,

A graphing calculator is required for some problems or parts of problems.

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Work for problem 1(a)

Rate of accumalation of snow = $7te^{\cos t}$

Accumulation at 6 A.M = $\int_0^6 (7te^{\cos t})\,dt$

≈ 142.275 ft^3

Work for problem 1(b)

Volume of snow at 8 A.M = $7te^{\cos t} - 108$

$\frac{dV}{dt} = (7t)(e^{\cos t} \cdot -\sin t) + (7)(e^{\cos t})$

$\frac{dV}{dt} = 7t(-e^{\cos t}\sin t) + 7e^{\cos t}$

$\frac{dV}{dt} = -7te^{\cos t}\sin t + 7e^{\cos t}$

At $t = 8$, $\frac{dV}{dt} \approx -41.8496$ ft^3/hr

Continue problem 1 on page 5

Work for problem 1(c)

$$h(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125t & \text{for } 6 \le t < 7 \\ 108t & \text{for } 7 \le t \le 9 \end{cases}$$

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Work for problem 1(d)

Total amount of snow falling from $0 \le t \le 9$

$= \int_0^9 (7te^{\cos t})\,dt \approx 367.33461\ ft^3$

From $6 \le t < 7$, Janet removed:

$\int_6^7 125\,dt = 125\ ft^3$

From $7 \le t \le 9$, Janet removed

$\int_7^9 108\,dt = 216\ ft^3$

So, at $t=9$, total snow $= (367.33461) - (125) - (216) \approx 26.335\ ft^3$

GO ON TO THE NEXT PAGE.

CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

1C1

A graphing calculator is required for some problems or parts of problems.

Work for problem 1(a)

$f(t)=7te^{\cos t}$

$\int_0^6 (7te^{\cos t})\,dt = 142.275 \text{ ft}^3$

Work for problem 1(b)

$f(8)=7(8)e^{\cos 8}$

$=48.417 \text{ ft}^2/h$

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Continue problem 1 on page 5.

Work for problem 1(c)

$$\int_6^7 125\,dt = 125$$

$$\int_7^9 108\,dt = 216$$

$$h(t)\begin{cases} 0, & 0 \le t < 6 \\ 125, & 6 \le t < 7 \\ 216, & 7 \le t \le 9 \end{cases}$$

Work for problem 1(d)

$$\int_0^9 (7te^{\cos t})\,dt = 367.335 \text{ ft}^2$$

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Question 1

Overview

This problem supplied two rate functions related to the amount of snow on Janet's driveway during a nine-hour period. One function f, given by $f(t) = 7te^{\cos t}$, measured in cubic feet per hour, models the rate of accumulation on the driveway for t between 0 and 9 hours after midnight. A second function, g, is a step function that gives the rate at which Janet removes snow from the driveway during this period. For part (a) students needed to use the definite integral $\int_0^6 f(t)\,dt$ to calculate the accumulation of snow on the driveway by 6 A.M. — integrating the rate of accumulation of snow over a time interval gives the net accumulation of snow during that time period. Part (b) asked for the rate of change of the volume of snow on the driveway at 8 A.M.; students needed to recognize this as the difference $f(8) - g(8)$ between the rate of accumulation and the rate of removal at time $t = 8$. Part (c) asked the students to recover a function h measuring the total amount of snow removed from the driveway for t between 0 and 9 hours after midnight. Students needed to integrate to obtain a piecewise-linear expression for h from the step function g. Part (d) asked for the amount of snow on the driveway at 9 A.M., which required students to compute the difference of two integrals, $\int_0^9 f(t)\,dt - \int_0^9 g(t)\,dt$.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's work is correct. In part (b) the student works with f', rather than f and g. The student's numeric answer is incorrect. In part (c) the student earned the first point for correctly identifying $h(t) = 0$ on the interval from 0 to 6. The second point was not earned since the student reports that the linear expression is $125t$. The student does not use the initial condition that $h(7) = 125$ and does not horizontally translate the linear expression, so the third point was not earned. In part (d) the student's work is correct.

Sample: 1C
Score: 4

The student earned 4 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student does not subtract $g(8)$ from the evaluation of $f(8)$. In part (c) the student earned the first point for correctly identifying $h(t) = 0$ on the interval from 0 to 6. The student presents constant functions for the other intervals and did not earn the other two points. In part (d) the student earned the point for the correct integral expression.

Question 2

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$.

Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

3 : $\begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \\ 1 : \text{meaning} \end{cases}$

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3 : $\begin{cases} 1 : \text{considers } P'(t) = 0 \\ 1 : \text{identifies candidates} \\ 1 : \text{answer with justification} \end{cases}$

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

Work for problem 2(a)

$$\frac{dE(6)}{dt} \approx \frac{E(7)-E(5)}{7-5}$$

$$= \frac{21-13}{2}$$

$$= \frac{8}{2}$$

$$= 4$$

about 400 entries per hour at $t=6$.

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Work for problem 2(b)

$$\frac{1}{8}\int_0^8 E(t)\,dt \approx \frac{1}{8}\left[\frac{1}{2}(2-0)(4+0)+\frac{1}{2}(5-2)(4+13)+\frac{1}{2}(7-5)(21+13)+\frac{1}{2}(8-7)(23+21)\right]$$

$$= \frac{1}{8}\left[\cancel{\tfrac{1}{2}}\cancel{(2)}(4)+\frac{1}{2}(3)(17)+\cancel{\tfrac{1}{2}}\cancel{(2)}(34)+\frac{1}{2}(1)(44)\right]$$

$$= \frac{1}{8}\left(4+\frac{51}{2}+34+22\right) = \frac{1}{8}\left(\frac{171}{2}\right) = \frac{171}{16} \text{ (or } 10.6875)$$

$\frac{1}{8}\int_0^8 E(t)\,dt$ is approximately 10.6875 [hundred entries].

$\frac{1}{8}\int_0^8 E(t)\,dt$ signifies the average value of the number of entries over the interval $0 \le t \le 8$ in hundreds

Continue problem 2 on page 7.

Work for problem 2(c)

# of processed entries = $\int_8^{12} p(t)\,dt$

$= \int_8^{12} (t^3-30t^2+298t-976)\,dt = 16$

∴ the # of unprocessed entries is $(E(8)-16)$ * in hundreds, which is $23-16=7$

$\boxed{\text{7 hundred entries had not yet been processed}}$

Work for problem 2(d)

when P(t) is at maximum, the rate of process is the fastest

→ P(t) @ local max, when $P'(t)=0$ and $P''(t)<0$

$P'(t) = 3t^2-60t+298=0 \rightarrow t=9.1835$ or $t=10.8165$

$P''(t) = 6t-60$

$P''(9.1835) = 6(9.1835)-60<0$

$P''(10.8165) = 6(10.8165)-60>0$ ∴ $t=10.8165$ is not valid

→ P(t) at end points (aka $t=8$ or $t=12$)

$P(8)=0$

$P(12)=8$ $P(12) > P(9.1835)$

and $P(9.1835) = 5.08866$

∴ P(t) is at maximum at $t=12$.

$\boxed{\text{at midnight, the entries are being processed most quickly}}$

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t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

Work for problem 2(a)

$$\frac{E(7) - E(5)}{7-5} = \frac{21-13}{2} = 4 \text{ hundreds/hr}$$

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Work for problem 2(b)

$$\frac{1}{8}\int_0^8 E(t)\,dt$$

$$= \frac{1}{8}\left[\frac{(0+4)(2-0)}{2} + \frac{(4+13)(5-2)}{2} + \frac{(13+21)(7-5)}{2} + \frac{(21+23)(8-7)}{2}\right]$$

$\approx$ 10.688 hundreds of entries

The average numbers of entries is about 10.688 hundreds

Continue problem 2 on page 7.

Work for problem 2(c)

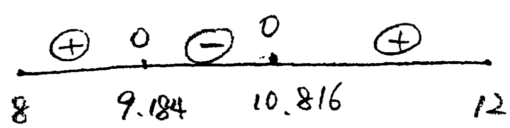
$\int_8^{12} P(t)\,dt = 16$ hundreds

$23 - 16 = 7$ hundreds

Work for problem 2(d)

$P'(t) = 3t^2 - 60t + 298$

$P'(t) = 0 \Rightarrow t \approx 9.184 \quad t \approx 10.816$



At $t = 9.184$, the entries were being processed most quickly because $P'(t) = 0$ at $t = 9.184$ and $P'(t)$ changes from positive to negative, which means local maximum occurs at $t = 9.184$, which was the time that entries processed most quickly

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

Work for problem 2(a)

$$\frac{E(7)-E(5)}{7-5} = \frac{21-13}{7-5} = 4$$

$\boxed{\approx 400 \text{ entries per hour}}$

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Work for problem 2(b)

$$= \frac{(2-0)\left(\frac{4-0}{2}\right)+(5-2)\left(\frac{13-4}{2}\right)+(7-5)\left(\frac{21-13}{2}\right)+(8-7)\left(\frac{23-21}{2}\right)}{8}$$

$\boxed{\approx 331.25 \text{ entries per hour}}$

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average rate of entries per hour.

Continue problem 2 on page 7.

Work for problem 2(c)

$23 - \int_8^{12} P(t)\,dt = \boxed{700 \text{ entries}}$

Work for problem 2(d)

$t=8$ $P(8)=0$
$t=9.1835$ $P(9.1835)=5.087$
$t=12$ $P(12)=8$

At $t=12$, the entries were being processed the quickest because the rate of change is largest there.

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Question 2

Overview

This problem involved a zoo's contest to name a baby elephant. Students were presented with a table of values indicating the number of entries $E(t)$, measured in hundreds, received in a special box and recorded at various times t during an eight-hour period. Part (a) asked for an estimate of the rate of deposit of entries into the box at time $t = 6$. Students needed to recognize this rate to be the derivative value $E'(6)$. Since $t = 6$ falls between the time values specified in the table, students needed to calculate the average rate of change of E across the smallest time subinterval from the table that brackets $t = 6$. Part (b) asked for an approximation to $\frac{1}{8}\int_0^8 E(t)\,dt$ using a trapezoidal sum and the subintervals of $[0, 8]$ indicated by the data in the table. Students were further asked to interpret this expression in context, with the expectation that they would recognize that it gives the average number of hundreds of entries in the box during the eight-hour period. In part (c) a function P was supplied that models the rate at which entries from the box were processed, by the hundred, during a four-hour period ($8 \le t \le 12$) that began after all entries had been received. This part asked for the number of entries that remained to be processed after the four hours. Students needed to recognize that the number of entries processed is given by $\int_8^{12} P(t)\,dt,$ so that the number remaining to be processed, in hundreds of entries, is given by the difference between the total number of entries in the box, $E(8)$, as given by the table, and the value of this integral. Part (d) cited the model $P(t)$ introduced in the previous part and asked for the time at which the entries were being processed most quickly. Students should have recognized this as asking for the time corresponding to the maximum value of $P(t)$ on the interval $8 \le t \le 12$ and applied a standard process for optimization on a closed interval.

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student sets up a correct difference quotient based on the values in the table and correctly evaluates for the numerical answer. In part (b) the student sets up a correct trapezoidal sum and evaluates it based on the data in the table to obtain a correct approximation. The student did not earn the third point in part (b) because the meaning given does not address the time interval over which the average was computed. In part (c) the student earned both points. The first point was earned for correctly providing the definite integral that represents the number of hundreds of entries processed between 8 P.M. and midnight. The second point was earned for subtracting that value from the initial condition of 23 hundred entries in the box at 8 P.M. to obtain the answer. In part (d) the student earned the first point for setting $P'(t) = 0$. The student does not consider the endpoints as candidates, so no additional points were earned.

Question 2 (continued)

Sample: 2C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student sets up a correct difference quotient based on the values in the table and correctly evaluates for the numerical answer. In part (b) the student subtracts the function values at endpoints of the subintervals rather than adding them. The student interprets the integral expression as an average rate rather than an average number. In part (c) the student's work is correct. "700" was accepted because of the units in this question. In part (d) the student never considers $P'(t)$, so no points were earned.

Question 3

A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.
 (i) The two values of t when that occurs
 (ii) The slopes of the lines tangent to the particle's path at that point
 (iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 2.828$ meters per second	1 : answer
(b) $x'(t) = 2t - 4$ Distance $= \int_0^4 \sqrt{(2t-4)^2 + \left(te^{t-3} - 1\right)^2}\, dt = 11.587$ or 11.588 meters	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ when $te^{t-3} - 1 = 0$ and $2t - 4 \ne 0$ This occurs at $t = 2.20794$. Since $x'(2.20794) > 0$, the particle is moving toward the right at time $t = 2.207$ or 2.208.	$3 : \begin{cases} 1 : \text{considers } \frac{dy}{dx} = 0 \\ 1 : t = 2.207 \text{ or } 2.208 \\ 1 : \text{direction of motion with reason} \end{cases}$
(d) $x(t) = 5$ at $t = 1$ and $t = 3$ At time $t = 1$, the slope is $\left.\frac{dy}{dx}\right\|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=1} = 0.432$. At time $t = 3$, the slope is $\left.\frac{dy}{dx}\right\|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=3} = 1$. $y(1) = y(3) = 3 + \frac{1}{e} + \int_2^3 \frac{dy}{dt}\, dt = 4$	$3 : \begin{cases} 1 : t = 1 \text{ and } t = 3 \\ 1 : \text{slopes} \\ 1 : y\text{-coordinate} \end{cases}$

Work for problem 3(a)

a. Speed $= \cancel{\int} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ $\quad \frac{dx}{dt} = 2t-4$

Speed $= \sqrt{(2t-4)^2 + (te^{t-3}-1)^2} \; \Big|_{t=3} \rightarrow$ 2.828 m/s

Work for problem 3(b)

total distance $= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^4 \sqrt{(2t-4)^2+(te^{t-3}-1)^2}\, dt$

$=$ 11.588 meters

Continue problem 3 on page 9.

Work for problem 3(c)

Find where $\frac{dy}{dx}=0$.

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{te^{t-3}-1}{2t-4}$

$0=\frac{te^{t-3}-1}{2t-4}$, $t=2.208$

If $\frac{dx}{dt}$ is positive, motion is to right, if negative, motion is to left.

$\frac{dx}{dt}=2t-4$, $|t=2.208$ $=0.416$, therefore the motion is to the right.

Work for problem 3(d)

i. $x(t)=5$, $5=t^2-4t+8$, $t=1$ and $t=3$

ii. at $t=1$: $\frac{dy/dt}{dx/dt}=dy/dx=\frac{te^{t-3}-1}{2t-4}$, $|t=1 \rightarrow$ $\frac{dy}{dx}$ at $t=1$ is 0.432

at $t=3$: $\frac{te^{t-3}-1}{2t-4}$, $|t=3 \rightarrow$ $\frac{dy}{dx}$ at $t=3$ is 1

iii. $y(3)=y(2)+\int_2^3 \frac{dy}{dt}$

$y(3)=\left(3+\frac{1}{e}\right)+\int_2^3 te^{t-3}-1\,dt$

$y(3)=4=y(1)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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3B1

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Work for problem 3(a)

speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt} = 2t - 4$ $\quad \frac{dy}{dt} = te^{t-3} - 1$, $t = 3$

speed $= \sqrt{(2)^2 + (3e^0 - 1)^2} =$

$\sqrt{8} = \boxed{2.8284 \ \frac{\text{units}}{s}}$

Work for problem 3(b)

distance $= \int |v(t)| =$

$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

distance $= \int_0^4 \sqrt{(2t-4)^2 + (te^{t-3} - 1)}$ $=$

$\boxed{11.5877 \text{ units}}$

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Continue problem 3 on page 9.

Work for problem 3(c)

$\frac{dy}{dx} = 0$ when horizontal

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{te^{t-3} - 1}{2t-4} = 0$

$\boxed{t = 2.20794}$

The particle is moving to the left because $\frac{dy}{dx}$ changes from positive to negative.

Work for problem 3(d)

$(5, y)$

a) $x(t) = t^2 - 4t + 8 = 5$

$t^2 - 4t + 3 = 0 \quad (t-1)(t-3)$

$\boxed{t = 1, \ t = 3}$

b) $\frac{dy}{dx} = \frac{te^{\cancel{t}-3} - 1}{2\cancel{x} - 4}$, $x = 5$ 3

$= \boxed{\frac{5e^2 - 1}{6}}$

$\frac{dy}{dx} = \cancel{\frac{1}{2e^2}}$

$\frac{dy}{dt} = te^{t-3} - 1$

c) $y(2) = 3 + \frac{1}{e}$

$\cancel{x} = 5 \quad \int_2^5 \frac{dy}{dt} = \boxed{\int_2^5 te^{t-3} - 1 + 3 + \frac{7}{e}}$

END OF PART ~~A OF SECTION II~~

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$\int dy = \int_0^3 (te^{t-3} - 1)\,dt$

$y = -.95$

$x'(t) = 2t-4$

speed $= \sqrt{x'(t)^2 + y'(t)^2}$

$t=3$ $\sqrt{(2t-4)^2 + (te^{t-3}-1)^2} =$

$= 4 + 4 = 8$ m/s

Work for problem 3(b)

$x(4) = 16 - 16 + 8 = 8$ m

$x'(t) = 2t - 4$

$v(t) = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{te^{t-3}-1}{2t-4}$

$\int_0^4 \frac{te^{t-3}-1}{2t-4}\,dt$

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Continue problem 3 on page 9.

Work for problem 3(c)

$$\frac{dy}{dx} = \frac{te^{t-3}-1}{2t-4} = 0$$

$t=0$ right b/c x is positive

Work for problem 3(d)

$t^2-4t+8=5$

$t^2-4t+3=0$

$(t-1)(t-3)$

$t=1,3$

$t=1 \quad m = \frac{dy}{dx} = \frac{te^{t-3}-1}{2t-4} = \frac{e^{-2}-1}{-2}$

$t=3 \quad m = \frac{2}{2} = 1 \qquad y(2) = 3+\frac{1}{e}$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Question 3

Overview

This problem described the path of a particle whose motion is described by $(x(t),\ y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ satisfies $\frac{dy}{dt} = te^{t-3} - 1$. Part (a) asked for the speed of the particle at time $t = 3$ seconds. Part (b) asked for the total distance traveled by the particle for $0 \le t \le 4$ seconds. This is found by integrating $\sqrt{(x'(t))^2 + (y'(t))^2}$ over the interval $0 \le t \le 4$. Part (c) asked for the time t, $0 \le t \le 4$, at which the line tangent to the particle's path is horizontal and whether the particle's direction of motion is toward the left or toward the right at that time. Students needed to solve $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ for t and determine the sign of $\frac{dx}{dt}$ at this time to establish the left-to-right direction of motion. In part (d) it was given that there is a point with x-coordinate 5 through which the particle passes twice. Students were asked for (i) the two values of t when that occurs, (ii) the slopes of the lines tangent to the particle's path at that point and (iii) the y-coordinate of that point, given that $y(2) = 3 + \frac{1}{e}$. After solving $x(t) = 5$ for $t = 1$ and $t = 3$, the slopes can be found by evaluating $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ at each value of t and the y-coordinate by evaluating $y(3) = y(2) + \int_2^3 \frac{dy}{dt}\,dt$ (or the corresponding expression for $t = 1$).

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

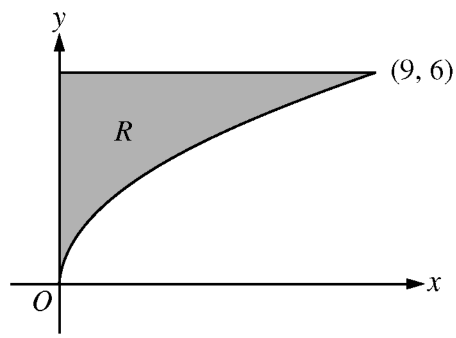
The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student considers $\frac{dy}{dx} = 0$ and earned the first point. The student correctly solves the equation to find the time at which the line tangent to the path of the particle is horizontal and earned the second point. The student incorrectly reasons that the motion of the particle at the t-value presented can be determined from $\frac{dy}{dx}$ and did not earn the third point. In part (i) of part (d), the student correctly solves $x(t) = 5$ for the two values $t = 1$ and $t = 3$ where the x-coordinate is 5 and so earned the point. In part (ii) of part (d), the student does not evaluate $\frac{dy}{dx}$ at $t = 1$ and $t = 3$ and did not earn the point. In part (iii) of part (d), the student presents an expression and numerical evaluation for the y-coordinate at an incorrect t-value and did not earn the point.

Question 3 (continued)

Sample: 3C
Score: 3

The student earned 3 points: no points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student has an incorrect evaluation of the expression for the speed at $t = 3$ and did not earn the point. In part (b) the student presents an incorrect integrand in the definite integral and did not earn any points. In part (c) the student considers $\frac{dy}{dx} = 0$ and earned the first point. The student does not solve the equation and did not earn additional points in part (c). In part (i) of part (d), the student correctly solves $x(t) = 5$ for the two values $t = 1$ and $t = 3$ where the x-coordinate is 5 and so earned the point. In part (ii) of part (d), the student uses the chain rule to correctly evaluate $\frac{dy}{dx}$ to find the slopes of the lines tangent to the path of the particle at $t = 1$ and $t = 3$ and so earned the point. In part (iii) of part (d), the student does not present any work.

Question 4



Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(a) Area $= \int_0^9 \left(6 - 2\sqrt{x}\right) dx = \left(6x - \frac{4}{3}x^{3/2}\right)\Big|_{x=0}^{x=9} = 18$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^9 \left(\left(7 - 2\sqrt{x}\right)^2 - (7-6)^2\right) dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) Solving $y = 2\sqrt{x}$ for x yields $x = \dfrac{y^2}{4}$.

Each rectangular cross section has area $\left(3\dfrac{y^2}{4}\right)\left(\dfrac{y^2}{4}\right) = \dfrac{3}{16}y^4$.

Volume $= \int_0^6 \dfrac{3}{16}y^4 \, dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

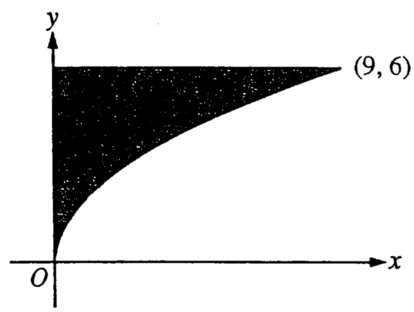
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

$R = \int_0^9 6\,dx - \int_0^9 2\sqrt{x}\,dx$

$9^{3/2} = 27$

$54 - 2\left[\frac{2x^{3/2}}{3}\Big|_0^9\right]$

$\frac{27}{3} = 9$

$54 - 2[18 - 0]$

$54 - 36$

$\boxed{\text{Area} = 18}$ units^2

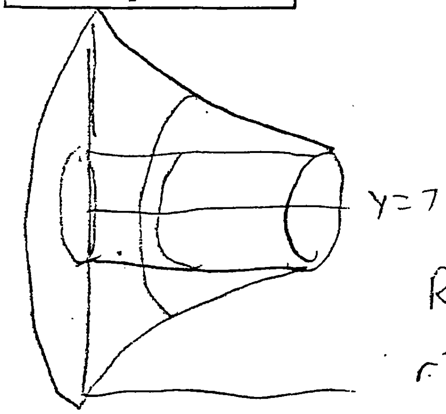
Continue problem 4 on page 1

NO CALCULATOR ALLOWED

Work for problem 4(b)

Vertical Washers $\cancel{\pi(R^2-r^2)}$ dx = thickness

$(\pi R^2 - \pi r^2)\,dx$



$\pi(R^2-r^2)$

$R = 7 - 2\sqrt{x}$

$r = 7-6$

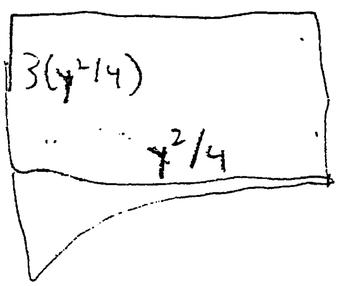
$r = 1$

$\boxed{\text{Volume} = \pi \int_0^9 \left[(7-2\sqrt{x})^2 - 1\right] dx}$

Work for problem 4(c)

$2\sqrt{x} = y \quad 4\cdot x = y^2$

$x = \frac{y^2}{4}$



$\int_0^6 \left[\frac{y^2}{4} \cdot \frac{3y^2}{4}\right] dy$

Volume = of Solid . $\boxed{\int_0^6 \frac{3y^4}{16}\,dy}$

GO ON TO THE NEXT PAGE.

4B1

NO CALCULATOR ALLOWED

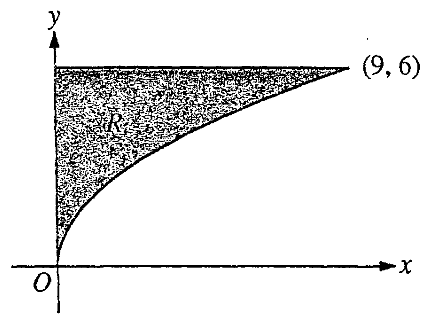
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$$R = \int_0^9 (6-2\sqrt{x})^2\,dx = \int_0^9 (36 - 24x^{1/2} + 4x)\,dx$$

$$= \left[36x - 16x^{3/2} + 2x^2\right]_0^9$$

$$= (324 - 432 + 162) - (0)$$

$$= (-108 + 162)$$

$$= \boxed{54 \text{ un.}^2}$$

$36 \times 9 = 324$

$81 \times 9 = 729$

$\sqrt{729} = \sqrt{3\,3\,3\,3\,3\,3} = 27$

$27 \times 16 = 162 + 270 = 432$

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Continue problem 4 on page 11.

4B2

NO CALCULATOR ALLOWED

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Work for problem 4(b)

$$\boxed{V = \pi\int_0^9 \left[(7-2\sqrt{x})^2-(1)^2\right]dx}$$

Work for problem 4(c)

$(y)^2 = (2\sqrt{x})^2$

$y^2 = 4x$

$x = \frac{y^2}{4}$

Height of rectangle $= 3\left(\frac{y^2}{4}\right)$

Base $= \frac{y^2}{4}$

Area of rectangle $= 3\left(\frac{y^2}{4}\right)\left(\frac{y^2}{4}\right)$

$= \frac{3y^4}{16}$

$$\boxed{\text{Volume} = \int_0^6 \left(\frac{3y^4}{16}\right)dy}$$

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GO ON TO THE NEXT PAGE.

4C1

NO CALCULATOR ALLOWED

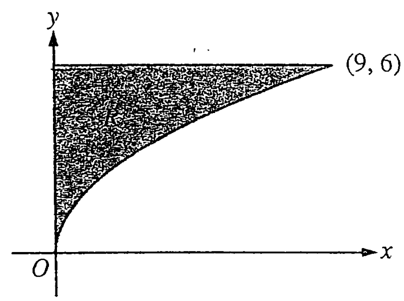
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$2(x)^{1/2 - 2}$

$\cancel{2} \cdot \frac{1}{\cancel{2}} x^{-\frac{1}{2}}$

$\int_0^9 6 - 2\sqrt{x}\, dx$

$0 - x^{-\frac{1}{2}} \Big|_0^9 = \frac{1}{\sqrt{x}} \Big|_0^9 = \frac{1}{\sqrt{9}} - \frac{1}{\sqrt{0}} = \boxed{\frac{1}{3}}$

$\boxed{R = \frac{1}{3}}$

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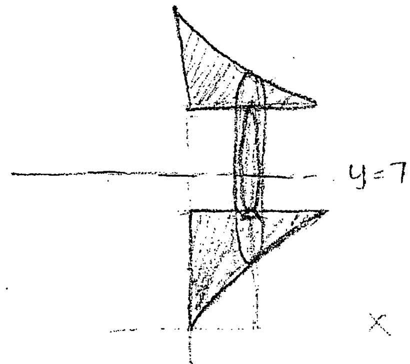
Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)



$\pi\int_0^9 (2\sqrt{x}-7)^2-(6-7)^2\,dx$

-1

$\pi\int_0^9 (4x+49-28\sqrt{x})-(1)\,dx$

$\pi\int_0^9 4x+48-28\sqrt{x}\,dx$

$(x)^{\frac{1}{2}+2}$ $\frac{3}{2}$

$\pi\left(2x^2+48x-28\cdot\frac{2}{3}(x)^{3/2}\right)\Big|_0^9$

$\sqrt{9}\sqrt{9}\sqrt{9}$ $3\cdot3\cdot3$

$\pi\left(2\cdot 81+48\cdot 9-\frac{56}{3}(9)^{3/2}\right)$

$\sqrt{9^3}$ 9 27

162
422
584
−494
90

56
+4
494

$\pi(90)=\boxed{90\pi}$

Work for problem 4(c)

$\pi\int_0^6 3\left((2\sqrt{x})^2-(6)^2\right)dx$

$\pi\int_0^6 3(4x-36)\,dx$

$\boxed{\pi\int_0^6 (12x-108)\,dx}$

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GO ON TO THE NEXT PAGE.

Question 4

Overview

In this problem students were given the graph of a region R bounded on the left by the y-axis, below by the curve $y = 2\sqrt{x}$, and above by the line $y = 6$. In part (a) students were asked to find the area of R, requiring an appropriate integral (or difference of integrals), antiderivative and evaluation. Part (b) asked for an integral expression that gives the volume of the solid obtained by revolving R about the line $y = 7$. This is found by integrating cross-sectional areas that correspond to washers with outer radius $7 - 2\sqrt{x}$ and inner radius 1, where $0 \le x \le 9$. Part (c) asked for an integral expression for the volume of a solid whose base is the region R and whose cross sections perpendicular to the y-axis are rectangles of height three times the lengths of their bases in R. Students needed to find the cross-sectional area function in terms of y and use this as the integrand in an integral with lower limit $y = 0$ and upper limit $y = 6$.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: no points in part (a), 3 points in part (b), and 3 points in part (c). In part (a) the integrand is shown as the square of the expected integrand, so the student was not eligible for any points. In parts (b) and (c), the student's work is correct.

Sample: 4C
Score: 4

The student earned 4 points: 1 point in part (a), 3 points in part (b), and no points in part (c). In part (a) the student's integrand is correct, but the antiderivative is incorrect; the student differentiated rather than antidifferentiated. No additional points were earned in part (a). In part (b) the student presents an integral in the first line of the solution that earned all 3 points. The student works with the integral, making no errors in lines two and three, and finding an antiderivative in line four. The student's work in lines four and beyond was not evaluated since the question asked for an integral expression only, not for the value of the integral. In part (c) the student's integral was not eligible for any points.

Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim\limits_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a) $$f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right|_{(1,\,0)}\right) \cdot \Delta x$$
$$= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$
$$f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\right) \cdot \Delta x$$
$$\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim\limits_{x \to 1} f(x) = 0 = \lim\limits_{x \to 1}\left(x^3 - 1\right)$ and we may apply L'Hospital's Rule.

$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim\limits_{x \to 1} f'(x)}{\lim\limits_{x \to 1} 3x^2} = \frac{1}{3}$$

$2 : \begin{cases} 1 : \text{use of L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(c) $$\frac{dy}{dx} = 1 - y$$
$$\int \frac{1}{1 - y}\,dy = \int 1\,dx$$
$$-\ln|1 - y| = x + C$$
$$-\ln 1 = 1 + C \Rightarrow C = -1$$
$$\ln|1 - y| = 1 - x$$
$$|1 - y| = e^{1-x}$$
$$f(x) = 1 - e^{1-x}$$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

Work for problem 5(a)

Using this table, we calculate $\frac{dy}{dx}$, allowing us to approximate $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$

x	y	dy / dx	Δy	Δx
1	0	1 − 0 = 1	−0.5	−0.5
0.5	−0.5	1 − (−0.5) = 1.5	−0.75	−0.5
0	−1.25	X	X	X

$$f(0) = y(0) \approx \boxed{-1.25}$$

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Work for problem 5(b)

We are given $f(1) = 0$, and $f'(x) = 1 - f(x)$

$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \frac{0}{0} = \text{Indeterminate}$$

Using L'Hopital's

$$\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \lim_{x \to 1} \frac{1 - f(x)}{3x^2} = \boxed{\frac{1}{3}}$$

Continue problem 5 on page 13.

Work for problem 5(c)

$$\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{1-y} = dx$$

$$\int \frac{dy}{1-y} = \int dx$$

$$-\ln|1-y| = x + c$$

$$\ln|1-y| = -x + c$$

$$e^{\ln|1-y|} = Ae^{-x}$$

$$1 - y = Ae^{-x}$$

$$y = 1 - Ae^{-x}$$

$$y(1) = 1 - \frac{A}{e} = 0$$

$$1 = \frac{A}{e} \Rightarrow A = e$$

$$y = 1 - e \cdot e^{-x} \Rightarrow \boxed{y = 1 - e^{-x+1}}$$

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Do not write beyond this border.

NO CALCULATOR ALLOWED

Work for problem 5(a)

x	y	Δx	m	Δy
1	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{4}$
0	$-\frac{5}{4}$			

$f(0) \approx -\frac{5}{4}$

Work for problem 5(b)

$$\lim_{x\to 1} \frac{f(x)}{x^3-1} \rightarrow \lim_{x\to 1} \frac{f'(x)}{3x^2} \rightarrow \lim_{x\to 1} \frac{f''(x)}{6x}$$

$$\rightarrow \frac{-1}{6x} = \frac{-1}{6(1)} = -\frac{1}{6}$$

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Continue problem 5 on page 1

Work for problem 5(c)

$f(1)=0$

$\frac{dy}{dx} = 1-y$

$\frac{dy}{1-y} = dx$

$\int \frac{dy}{1-y} = \int dx$

$\ln|1-y| = x + C$

$e^{\ln|1-y|} = e^x + e^C$

$1-y = Ae^x$

$y = 1-Ae^x$

$0 = 1-Ae^1$

$1 = Ae$

$A = \frac{1}{e}$

$y = 1-(\frac{1}{e})(e^x)$

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

Work for problem 5(a)

$\frac{dy}{dx} = 1 - y \quad f(1)=0 \quad f(x)<1$

$x=1 \to f(1)=0$

$f(.5) \approx f(1) + (-.5)(f'(1))$

$= 0 + (-.5)(1-0)$

$= -.5$

$f(0) \approx f(.5) + (-.5)(f'(.5))$

$= -.5 + (-.5)(1.5)$

$= -.5 + -.75$

$\approx \boxed{-1.25}$

Work for problem 5(b)

$\lim_{x \to 1} \frac{f(x)}{x^3-1}$

l'hospital to the rescue!

$\lim_{x \to 1} \frac{1-y}{3x^2} = \frac{1-f(1)}{3(1)^2}$

$= \frac{1-0}{3}$

$= \boxed{1/3}$

Do not write beyond this border.

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

Work for problem 5(c)

dont forget +C

$\frac{dy}{dx} = 1-y$

$\int dy = \int dx - y\,dx$

$y = x - \int y\,dx$

$(xy - \int x\,dy)$

$\int \frac{dy}{1-y} = \int dx$

$\ln|1-y| = x$

$1-y = e^x + C$

$y = -e^x - 1 + C$

$f(1)=0 \Rightarrow 0 = -e^1 + C$

$C = e$

$\boxed{y = -e^x + e}$

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Question 5

Overview

This problem presented the differential equation $\frac{dy}{dx} = 1 - y$ with a particular solution $y = f(x)$ satisfying $f(1) = 0$. It was also given that $f(x) < 1$ for all values of x. Part (a) asked the students to use Euler's method with two steps of equal size to approximate $f(0)$. Part (b) asked for the evaluation of $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$, anticipating that students would recognize an invitation to apply L'Hospital's Rule. Part (c) asked for the particular solution $y = f(x)$ with initial condition $f(1) = 0$. Students should have used the method of separation of variables.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the student's work is correct. In part (b) the student does not justify the use of L'Hospital's Rule and applies it too many times. In this particular case, the student moves beyond the first derivative and declares an incorrect answer. In part (c) the student earned the separation, constant of integration, and initial condition points. The final answer for $y = f(x)$ is consistent with the student's antiderivative error (missing a factor of -1) and earned the point for solving for y.

Sample: 5C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student earned the answer point but does not justify the use of L'Hospital's Rule. In part (c) the student earned the separation point. The student has an incorrect antiderivative and incorrectly applies the constant of integration. As a result, no additional points were earned.

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\, dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \cdots$

$f(x) = -\dfrac{1}{2} + \dfrac{x^2}{4!} - \dfrac{x^4}{6!} + \cdots + (-1)^{n+1} \dfrac{x^{2n}}{(2n+2)!} + \cdots$

$3 : \begin{cases} 1 : \text{terms for } \cos x \\ 2 : \text{terms for } f \\ \quad 1 : \text{first three terms} \\ \quad 1 : \text{general term} \end{cases}$

(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$.

$\dfrac{f''(0)}{2!} = \dfrac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \dfrac{1}{12}$.

Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.

$2 : \begin{cases} 1 : \text{determines } f'(0) \\ 1 : \text{answer with reason} \end{cases}$

(c) $P_5(x) = 1 - \dfrac{x}{2} + \dfrac{x^3}{3 \cdot 4!} - \dfrac{x^5}{5 \cdot 6!}$

$2 : \begin{cases} 1 : \text{two correct terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $g(1) \approx 1 - \dfrac{1}{2} + \dfrac{1}{3 \cdot 4!} = \dfrac{37}{72}$

Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \dfrac{37}{72} \right| < \dfrac{1}{5 \cdot 6!} < \dfrac{1}{6!}$.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{explanation} \end{cases}$

6A,

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Work for problem 6(a)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$$

First three nonzero terms: $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

general term = $\frac{(-1)^n (x)^{2n}}{(2n)!}$

$f(x) = \frac{\cos x - 1}{x^2}, x \neq 0$

First nonzero terms = $-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!}$

general term = $\frac{(-1)^{n+1} (x)^{2n}}{(2n+2)!}$

Work for problem 6(b)

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x)^{2n}}{(2n+2)!} = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \ldots$$

$f'(x) = \frac{2x}{4!} - \frac{4x^3}{6!} + \ldots = 0$ $\quad f'(x) = x\left(\frac{2}{4!} - \frac{4x^2}{6!} + \ldots\right)$

$x = 0$

$f''(x) = \frac{2}{4!} - \frac{12x^2}{6!} + \ldots$

$f''(0) = \frac{2}{4!} > 0$ $\cup$

$\therefore \exists$ relative min at $x = 0$ by the second derivative test

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Continue problem 6 on page 15.

6A₂

Work for problem 6(c)

$\sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!}(x)^n$ $g(x) \approx P_5(x) = 1 - \frac{1}{2}x + \frac{2}{4!3!}x^3 - \frac{24}{6!5!}x^5$

$g(0) = 1$

$g'(0) = -\frac{1}{2}$ $g'(x) = f(x)$

$g''(0) = 0$ $g''(x) = f'(x)$

$g'''(0) = \frac{2}{4!}$ $g'''(x) = f''(x)$

$g^4(0) = 0$ $g^4(x) = f'''(x) = \frac{-24x}{6!} + \ldots$

$g^5(0) = -\frac{24}{6!}$ $g^5(x) = f^4(x) = -\frac{24}{6!}$

Work for problem 6(d)

$g(x) \approx P_3(x) = 1 - \frac{1}{2}x + \frac{2}{4!3!}x^3$ $R_3(x) = \frac{24}{6!5!}x^5$

$g(1) \approx P_3(1) = 1 - \frac{1}{2}(1) + \frac{2}{4!3!}(1)^3$ $R_3(1) = \frac{24}{6!5!}$

$1 - \frac{1}{2} + \frac{2}{4!3!}$ $\begin{array}{r} 24 \\ \times 6 \\ \hline 144 \end{array}$ $= \frac{4!}{6!5!}$ $\begin{array}{r} 720 \\ \times 120 \\ \hline 144 \\ 720 \\ \hline 86400 \end{array}$

$\frac{1}{2} + \frac{2}{144}$ $= \frac{1}{(6!)5} < \frac{1}{6!}$

$\boxed{\frac{74}{144}}$

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NO CALCULATOR ALLOWED

6B,

Work for problem 6(a)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$f(x) = -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!}$

Work for problem 6(b)

$f'(x) = \frac{2x}{4!} - \frac{4x^3}{6!}$

$f'(0) = 0$

$f''(x) = \frac{2}{4!} - \frac{12x^2}{6!}$

$f''(0) = \frac{1}{12} > 0$

at $x=0$ $f(x)$ is at a minimum by second derivative test.

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

6B2

Work for problem 6(c)

$g(x) = 1 + \int_0^x f(t)\,dt$

$= 1 - \frac{x}{2!} + \frac{x^3}{(3)(4!)} - \frac{x^5}{(6!)(5)}$

Work for problem 6(d)

$g(1) \approx 1 - \frac{1}{2} + \frac{1}{(3)(4!)}$

→ to get error take next term and plug in for $x=1$

→ $\left|-\frac{1^5}{(6!)(5)}\right| < \frac{1}{6!}$

this method, of evaluating next term to find error, work because g about $x=0$ and evaluated at $x=1$ is an alternating series w/ terms that decrease in absolute value to 0.

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6C,

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Work for problem 6(a)

$\cos x \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad : \quad 1 - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^6}{6!} \cdots (-1)^n \frac{x^{2n}}{(2n)!}$

$f(x) = \frac{\cos x - 1}{x^2} \quad : \quad \frac{\left(1 - \frac{\cancel{x^2}}{2!} + \frac{x^{\cancel{8}\,2}}{4!} - \frac{x^{\cancel{6}\,4}}{6!} \cdots \frac{x^{2n}}{(2n)!}\right)}{x^2}$

$= -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} \cdots \frac{(-1)^{n+1}\, x^{2n}}{(2n)!}$

Work for problem 6(b)

$f'(0) = 0$

The Taylor series for f about $x = 0$

$f''(0) = \frac{(-1)^{4+1}\, 6^{2n}}{(2\cdot 4)!}$

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Continue problem 6 on page 15.

6C2

Work for problem 6(c)

Taylor for $g(x) = 1 + \int_0^x -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \frac{x^6}{8!} - \frac{x^8}{10!}$

$= 1 - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} - \frac{x^9}{10!}$

Work for problem 6(d)

g about $x = 0$, evaluated @ $x = 1$

Lagrange Error using third degree of $-\frac{x^5}{6!}$

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Question 6

Overview

This problem provided the function f defined by $f(x) = \frac{\cos x - 1}{x^2}$ for $x \neq 0$ and $f(0) = -\frac{1}{2}$. It was given that f has derivatives of all orders, and the function g is defined by $g(x) = 1 + \int_0^x f(t)\,dt$. Part (a) asked for the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Students were to use this with algebraic manipulation to find the first three nonzero terms and the general term of the Taylor series for f about $x = 0$. In part (b) students were asked to use the Taylor series for f about $x = 0$ to determine whether f has a relative maximum, relative minimum or neither at $x = 0$. From the series for f students can establish that $f'(0) = 0$ and $f''(0) = \frac{1}{4 \cdot 3}$ and resolve this issue using the Second Derivative Test. Part (c) asked for the fifth-degree Taylor polynomial for g about $x = 0$. In part (d) it was given that the Taylor series for g about $x = 0$ is an alternating series whose terms decrease in absolute value to 0. Students were asked to use the third-degree Taylor polynomial for g about $x = 0$ to estimate $g(1)$ and to explain why this estimate is within $\frac{1}{6!}$ of the actual value. The properties of the series for $g(1)$ allow us to bound the error in this approximation by the absolute value of the next term in the series.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student earned the second point. Although the first three terms for the series for $\cos x$ are correct, the student omits the general term. The first three terms of the series for f are correct, but the student also omits the general term. The student did not lose any points for the use of "=" in both cases. In part (b) the student incorrectly declares $f'(x)$ and $f''(x)$ to be polynomials and provides a correct solution without dealing with a series. The student earned just 1 point as a result. In parts (c) and (d), the student's work is correct. The student did not lose any points for an incorrect use of the equals sign in part (c).

Sample: 6C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student earned the first point for the first three terms and the general term of the series for $\cos x$. The first three terms of the series for f are correct, but the general term is incorrect. The student earned 1 of the possible points for the series for f. In part (b) the student earned the first point for determining $f'(0)$. In part (c) the student earned 1 point for the correct first two terms. In part (d) the student's work is incorrect.